1. Tracking changes in mathematical creative thinking of secondary school students which perform inquiry in mathematics.

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2. Abstract

The main purpose of the study was tracking changes in mathematical creative thinking of secondary school students which perform inquiry in mathematics. In the study, creativity is defined as a triad of fluency, flexibility and originality (as follows from Guilford (1967); Torrance (1962, 1968, 1979 & 1990)), also based on Leikin (2009), Silver & Cai (1996) and Braverman (2010) researches. Six students from 7^{th}-10^{th} grade classes participated in the study. Research instruments included: inquiry projects per se and individual semi-structured interviews. The scoring scheme for fluency, flexibility, and originality in interviews was adapted from Leikin (2009) and Braverman (2010).
We expected that, consistently with findings of other studies (e.g. Levav-Weinberg & Leikin, 2009, Braverman, 2010) this research will show that didactical tools (inquiry projects) lead to the development of creativity. The study developed a set of inquiry-based projects that can be used by teachers in mathematics classes of different levels (especially with high-achievers in mathematics). These projects can also be used in professional development courses for teachers of mathematics in order to intensify the implementation of inquiry-based approach in schools.

3. Description of the study

3.1 The main purpose of the study

The main purpose of the study is tracking changes in mathematical creative thinking of secondary school students which perform inquiry in mathematics.

3.2 Rationale

There is an opposition of two overriding goals of education (Wells, 1999, p.157). On the one hand, we need to ensure that the young absorb values, knowledge, and practices of their culture and society in order to become responsible citizens. On the other hand, creativity must be supported and encouraged in each student so that he/she could realize his/her unique potential. Unfortunately, the pressure to achieve the first goal is so overwhelming that in practice (at school) there is almost no time or opportunity left to attend to the second one (Wells, ibid).

When speaking about abilities (including mathematical abilities), an assumption is made that they differ among individuals. There are more gifted and less gifted students, and each of them can acquire knowledge from the school program. It should be remembered that the volume of knowledge students acquire varies (Krutetskii, 1968, p.6). Krutetskii also points out that the development of general abilities does not contradict the development of special abilities; therefore, there is no conflict between the above two goals. Teachers should foster all the abilities of a student, and pay attention to that student's major abilities that determine his/her professional propensities (Krutetskii, 1968, p.8).

The value and importance of creativity for each individual and humanity as a whole are undeniable. Creativity is a potential that enables the individual to solve life's problems and adapt to new circumstances, as well as to be actively engaged in changing the surrounding environment, life and future. In all the spheres of education, development of creativity must be of primary importance. However, it is often mentioned in the context of art and music, but less—in the context of mathematical education. The word *creativity* can be found in syllabuses and study programs. But due to the constraints imposed by the first goal, creativity fostering is seldom observed in mathematical classrooms.

The mission of school is to encourage and foster spontaneous creative impulses, internally motivated creative thinking and activity. The activity of this sort is natural to children (Urban, 1990, 1995). Most children are open to new experiences, seek novelty, ask people around them (family members or other individuals) questions, examine facts and things while actively interacting with them, especially in games. One of the teacher's main goals is to find a creative base in children later to be developed and refined with the help of various techniques (Urban, 1999).

One example of such technique is inquiry project. For the purpose of this study "*Inquiry Project* is defined as a set of mathematical problems with a specific theme that requires exploration, consideration of special cases, inductive thinking, generalization, conjecturing, examination of conjectures, proving or refuting them, and posing new problems based on the ideas of the original problem" (Braverman, 2010, p.2).

The main considerations that justify the importance of **Inquiry Projects** are (Braverman, 2010):

1. The process of finding a creative solution and/or posing a new problem very closely resembles the work of a mathematician. Therefore, already at school, a child can get an idea of what the work of a mathematician (or any other professional) is like. Undoubtedly, this can seriously influence the choice of one's future profession.
2. Usually great discoveries are made by mathematicians when they are 22–26 years old. Therefore, the suggestion is made that teaching scientific analysis techniques to secondary school students may be very promising. Moreover, as a result of this natural and internally motivated immersion into the creative process, divergent and convergent intellectual properties of mathematical thinking come into play, and can give impetus to a child's further development.
3. As part of an inquiry process, a child acquires additional knowledge, and advanced skills and abilities in many sub-disciplines of mathematics, which in itself is a valuable gain.
4. Finding a creative solution and/or posing a problem stimulate mathematical abilities and motivation to learn. (However, the level of a student's motivation depends also on the teacher (De Bono, 1976, p.148).)
5. Common classroom competition contributes to the development of creativity; yet it often causes social and psychological conflicts. Contrary to this, in the individual studying process (process of investigation or inquiry), students make comparisons only with their own achievements

To explore students' creativity, this study also uses the notions of ***solution spaces*** and ***multiple-solution task*** (MST) suggested by Leikin (2007, 2009).

‘A **multiple-solution task (MST)** is an assignment, in which a student is explicitly required to solve a mathematical problem in different ways’ (Leikin, 2009, p.133).

1. Individual solution space is a set of solutions to a particular problem, which are suggested by an individual. There are two kinds of individual solution spaces:

Available solution spaces include solutions obtained without the help of others.

Potential solution spaces include solutions obtained with the help of others. This notion follows from the definition of *zone of proximal development* (Vygotsky, 1978).

2. Expert solution space is a set of solutions to a particular problem, which are suggested by an expert mathematician. There are two kinds of expert solution spaces:

Conventional solution space is a set of solutions included in the school curriculum.

Unconventional solution space is a set of solutions, not included in the school curriculum.

3. Collective solution spaces:

Collective space is a set of solutions produced by a group of individuals.

Expert solution space includes both the individual solution space and the collective solution space.

This study based on the study of Braverman (2010). The main purpose of his study was to explore how systematic implementation of inquiry projects in school mathematics develops students' creativity. In the study was found a connection between the systematic implementation of inquiry projects in secondary school mathematics and the development of students’ problem-solving fluency and flexibility, the independence of problem-solving and problem-posing creativity (fluency, flexibility and originality) from students’ achievements in curricular mathematics, defined four main types of changes in the students’ problem-solving creativity. Consistently with findings of other studies (e.g. Leikin, 2009, Levav-Weinberg & Leikin, 2009) the research shows that didactical tools allow development of problem-solving mental fluency and flexibility. Development of problem-solving originality is questionable and requires additional investigation, however, the data obtained in this study supports the hypothesis, proposed by Leikin (2009), that problem-solving originality may be a stable cognitive characteristic that depends on personal abilities. The study proves that the model for evaluation of

mathematical creativity by means of MST, is applicable to problem-posing situations.
The study demonstrates once again that the combination of quantitative and qualitative research tools is essential for better understanding of the effects of instructional innovations. In the study was showed practical ways to implement inquiry projects in secondary school mathematics, to create didactical and a-didactical situations in the classroom and to transform the inquiry project into a serious research (for secondary school student).

The study includes a set of inquiry-based problems that can be used by teachers in mathematics classes of different levels (especially with high-achievers in mathematics). These problems can also be used in professional development courses for teachers in order to intensify implementation of inquiry-based approach in school mathematics.

Below introduced an example of inquiry project given to the students:

In each of the expressions bellow choose "+" or "-"*among two consecutive numbers in the expression so that the sum will be equal to zero.*

a) $\pm 1 \pm 2 \pm 3 \pm ... \pm 12$

b) $\pm 1 \pm 2 \pm 3 \pm ... \pm 13$

c) $\pm 1 \pm 2 \pm 3 \pm ... \pm 14$

d) $\pm 1 \pm 2 \pm 3 \pm ... \pm 15$

e) $\pm 1 \pm 2 \pm 3 \pm ... \pm n$

Possible solutions: we have many different solutions for the given problem which can be produced by different grouping of the given numbers. For example, we can group them by "quarters" of consequent numbers (according to remainders of 4).

If we have $n=4k$ terms, it is possible with combination (+ — — +) for each "quarter" of consequent numbers (because of $k - (k + 1) - (k + 2) + (k + 3) = 0$):

$(+ 1 -2 - 3+4)+(+5- 6- 7+8)+(+9- 10- 11+12)=0$

If $n=4k+3$ terms, it is possible with combination (+ + —) for the first three numbers and with combination (+ — — +) for each "quarter" of consequent numbers:

(+ 1 +2 – 3)+(+4– 5– 6+7)+(+8– 9– 10+11)+(+12– 13– 14+15)=0

If *n=4k+2 or n=4k+1* it is impossible (because in both these cases sum of all numbers is an odd number).

In present study the inquiry projects will be more structured and complicated, for example the above problem may be continued as follows:

Put "+" or "-" in expression: **± *1* ± *2* ± *3* ±...± *n*** so as to receive number N; $1 \leq N \leq n$.
This problem is much more complicated than previous one.

3.3 Research Questions

1. Does ***flexibility*** of students' mathematical thinking change in the process of their performing of inquiry projects?
   1a. How flexible are the students when producing multiple solutions to different mathematical problems at different stages of the study?
   1b. How flexible are the students when posing mathematical problems at different stages of the study?
2. Does ***fluency*** of students' work on mathematical tasks change in the process of implementation of inquiry projects in mathematics classes?
   2a. How fluent are the students when producing multiple solutions to different mathematical problems at different stages of the study?
   2b. How fluent are the students when posing mathematical problems at different stages of the study?
3. Does ***originality*** of students' mathematical solutions change in the process of implementation of inquiry projects in mathematics classes?
   3a. How original are the students when producing multiple solutions to different mathematical problems at different stages of the study?
   3b. How original are the students when posing mathematical problems at different stages of the study?

3.4 Methodology

3.4.1 Population

Six students from 7th-10th grade classes participated in the study. Work on the projects was voluntary. We planned study with ten students but due to the difficulties some students (4 of 10 who started the school year) stopped working on the inquiry projects. Teacher's mathematical background and his ability to conduct inquiry projects with his students served as the criteria for choosing the teacher who will be an advisor. He has been working in the field of mathematical instruction for more than 30 years, has strong mathematical background and knowledge of curricular material; has experience in work with non-curricular material (for example, Olympiad problems).

3.4.2 Variables

As was mentioned above, in the study, creativity was defined as a triad of fluency, flexibility and originality for problem solving and for problem posing. Also changes in each student individual solution space were tracked.

3.4.3 Research Tools

Research instruments include: inquiry projects per se and individual semi-structured interviews. The scoring scheme for fluency, flexibility, and originality in interviews was adapted from Leikin (2009) and Braverman (2010). It was also based on the definitions of correctness and solvability of the posed problem used in the research of Silver & Cai (1996).

The purpose of the inquiry projects was to enhance the students' creative thinking by teaching them the following skills: finding multiple solutions, problem posing, proving and refuting conjectures, making generalizations, examining special cases, using non-curricular methods of problem solving, solving non-curricular problems.

The purpose of the interviews was to compare the students' individual (potential and available) solution spaces and problem-posing spaces, to measure students' fluency, flexibility, and originality in problem solving and problem posing.

Inquiry Projects

As defined above, an inquiry project is a set of mathematical problems with a specific theme that requires the study of special cases, inductive thinking, generalization, and posing new original problems based on the ideas of a given problem.

The purpose of the inquiry projects is to enhance the students' creative thinking by teaching them the following skills:

- finding multiple solutions
- problem posing
- proving and refuting conjectures
- making generalizations and examining special cases
- using non-curricular methods of solving problems
- solving non-curricular problems

The ***inquiry projects*** will be developed according to the following principles:

1. Each task includes several sub-tasks with a gradual increase in difficulty level.
2. The tasks are chosen from different branches of mathematics.
3. The tasks require the knowledge of school mathematics curriculum only.
4. The tasks made possible acquiring new curricular and non-curricular knowledge and application of unconventional solutions.
5. Each task allows posing new mathematical problems

Here inquiry projects were given to the students.

I. Series of natural numbers arranged in a special way.

II. Optimization: balls' problem.

III. What is behind the Ramsey Theorem? Solution of complex geometric problems using Ramsey Theory

IV. Solution of symmetrical inequalities using majorants

V. Markov Diophantine equations

VI. Fake coins problems and their expansions for problems with more fake coins lying following to each other: finding algorithms

Semi-structured Interviews

The purpose of the interviews was to compare the students' individual (potential and available) solution spaces and problem-posing spaces at different stages of the study. The semi-structured interviews study the impact of inquiry-based tasks on the development of students' creativity.

Table 1 shows Interview Prompts and Criteria for Replies Analysis.

Table 1: Interview Prompts and Criteria for Replies Analysis

Interview Task / Prompts	**Focus of Analysis**
1. Solve the problem.	***Fluency:*** *The quickness* of the start of the solution *Capability* to choose special cases for investigation; generalization ***Flexibility:*** Conventionality of the first suggested solution ***Originality:*** Ability to provide an unconventional solution
2. *If the problem is solved*: Can you find an additional solution to the problem? One more solution?	***Flexibility:*** Multiple solutions ***Fluency:*** The pace of change between the solutions or solution approaches
3. *If the student takes a wrong direction:* Can you think of a different approach to the solution? Are you convinced in	***Flexibility:*** Ability to change the approach to the solution

your solution?	
4. Can you think of a different problem related to this one? Can you pose such problem?	***Flexibility:*** Ability to change the focal point of the task ***Originality:*** Ability to suggest an unconventional task

Each interview includes three mathematical problems. In order to answer the research questions, conventional and unconventional tasks were chosen for the interviews. Each interview consists of two conventional word problems and one unconventional problem that had three or more possible solutions.

The interviews were conducted at three different stages. Interview A was conducted at the start of the school year; interview B was at the middle of the school year; and interview C was conducted at the end of the school year. Our intention is to study the development of students' creativity by comparing the results of interviews A, B, and C.

If necessary, the students received prompts for each task as presented in Table 1.

An interview example is presented below.

Interview B

a) Find as many as possible solutions to the following problems

b) For each of the given problems, pose a new problem based on a similar idea

Problem 1

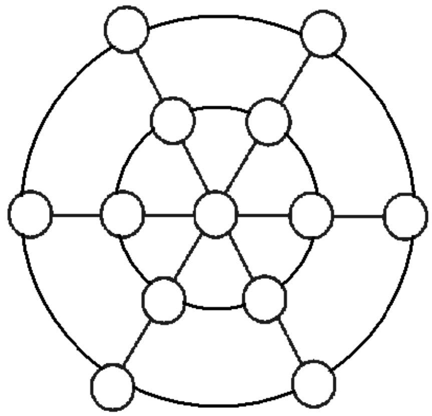
Rami has 3 times fewer stamps than Sara. Sara has 4 stamps more than Rami. How many stamps does each of them have?

Problem 2

David has three brothers. The first one is three years older than David, the second one is three years younger, and the third one is three times younger than David. The father is three times older than David. The total age of the five is 95 years. How old is each of them?

Problem 3

Put the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 & 12 into small circles so as to receive equal sums on each straight line and equal sums on the two big concentric circles (it is not necessary for the sums on the straight lines and on the circles to be equal).



Semi-structured Interviews Analysis

The scoring scheme of fluency, flexibility, and originality for interviews was adapted from Leikin (2009) and Braverman (2010).

PROBLEM SOLVING

<u>Fluency</u>

The student's fluency in an individual interview was determined by the number of solutions in the individual solution space. The student received 1 fluency point for each solution. Therefore, the student who solves the problem in more ways (or found a number of solutions to an unconventional problem) will receive more fluency points.

<u>Flexibility</u>

To evaluate flexibility, we use the term *groups of solutions for the MSTs* (Leikin, 2009). Two solutions belong to two different groups if their solution strategies are based on properties taken from different branches of mathematics. Table 2.a presents Flexibility Scoring Scheme for Problem Solving.

Table 2.a: Flexibility Scoring Scheme (Problem Solving, Interviews)

Problem Number		Interview A			Interview B			Interview C		
		1	2	3	1	2	3	1	2	3
First Solution		10	10	10	10	10		10	10	
For each subsequent solution (conventional problems)	Another representation of the same solution	0.1	0.1	0.1	0.1	0.1		0.1	0.1	
	Solution from the same group of solutions	1	1	1	1	1		1	1	

	Solution from another group of solutions	10	10	10	10	10		10	10	
For each unconventional problem	First solution			10			10			10
	Second/third (not last) solution			1			1			1
	Last solution			10			10			10

For the first solution of a conventional problem, the student received 10 flexibility points. The more the subsequent solution to a conventional problem differs from the previous solution, the more flexible it will be. Each subsequent solution to an unconventional problem yields 1 flexibility point, and the last solution yields 10 flexibility points (the student investigated all the cases). Groups of solutions were built using the students' solution spaces for interviews.

For example, if the total flexibility score for a solution space is 21.3, we know that it included 2 solutions that belonged to the same solution group (based on the same solution strategy), but differed in some essential characteristics; 1 solution that used a solution strategy different from 2 previous solutions (they were based on different solution strategies); and 3 solutions that repeated the previous ones (different representations of the above solutions).

The student's total flexibility score for a problem is the sum of student's flexibility scores for the solutions in the student's individual solution space.

Originality

We evaluated the originality of a solution based on its conventionality, the level of solution (Ervynck, 1991, pp. 44-45), and the solution conventionality according to the participants' learning history (absolute evaluation: for example, solving a problem with an equation was unconventional in Interview A, and conventional in Interviews B & C; solving a problem with a system of equations was unconventional in Interview A and partially unconventional in Interviews B & C). Table 2.b presents the Originality Scoring Scheme for Problem Solving for each problem in each interview.

Table 2.b: Originality Scoring Scheme (Problem Solving, Interviews)

For each solved problem	Conventional (learned) solution	0.1
	Partially unconventional solution (learned in a different context)	1
	Unconventional solution	10

For example, if the total originality score for the solution space of a problem is 12.3, we know that it includes 1 unconventional solution, 2 partially unconventional solutions and 3 conventional solutions.

The student's total originality score for a problem is the sum of student's originality score for the solutions in the student's individual solution space.

Creativity

The creativity (Cr_i) of a particular solution is the product of the solution's flexibility (Flx_i) and originality (Or_i): $Cr_i = Flx_i \cdot Or_i$

The total creativity score (Cr) on a MST is the sum of the creativity scores for each solution in the individual solution space of a problem: $Cr = \sum_{i=1}^{N} Flx_i \cdot Or_i$ (N = fluency).

For example, student produced 4 solutions for a particular problem:

The flexibility score for the first solution is 10 ($Flx_1 = 10$). If the student produced an original solution, he/she received 10 originality points ($Or_1 = 10$). In this case, the creativity score of the solution is: $Cr_1 = Flx_1 \cdot Or_1 = 10 \cdot 10 = 100$.

If Solution 2 belonged to the same group of solutions, the student received $Flx_2 = 1$ or $Flx_2 = 0.1$, $Or_2 = 10$ (the originality score is the same because the solution belongs to the same group), the creativity score for this solution is: $Cr_2 = Flx_2 \cdot Or_2 = 1 \cdot 10 = 10$ or $Cr_2 = Flx_2 \cdot Or_2 = 0.1 \cdot 10 = 1$.

If Solution 3 belonged to a <u>different</u> group than Solutions 1 & 2, the student received $Flx_3 = 10$. If the solution was not original: $Or_3 = 1$ (partially unconventional) or $Or_3 = 0.1$ (conventional).

So, we have $Cr_3 = Flx_3 \cdot Or_3 = 10 \cdot 1 = 10$ or $Cr_3 = Flx_3 \cdot Or_3 = 10 \cdot 0.1 = 1$

If Solution 4 was another representation of Solution 3, the student received $Flx_4 = 0.1$, $Or_4 = 1$ or $Or_4 = 0.1$. So, we have $Cr_4 = Flx_4 \cdot Or_4 = 0.1 \cdot 1 = 0.1$ or $Cr_4 = Flx_4 \cdot Or_4 = 0.1 \cdot 0.1 = 0.01$

The total creativity of a student in a concrete case may be equal:

$$Cr = \sum_{i=1}^{4} Cr_i = \sum_{i=1}^{4} Flx_i \cdot Or_i = Flx_1 \cdot Or_1 + Flx_2 \cdot Or_2 + Flx_3 \cdot Or_3 + Flx_4 \cdot Or_4 =$$
$$= 10 \cdot 10 + 1 \cdot 10 + 10 \cdot 0.1 + 0.1 \cdot 0.1 = 100 + 10 + 1 + 0.01 = 111.01$$

PROBLEM POSING

Fluency

The student's fluency in an individual interview was determined by the number of posed problems in the individual posing space. Therefore, the student, who posed more problems of his/her own, received more fluency points.

Flexibility

To evaluate flexibility, we examined the changes the student will make in a given problem. For changing only the story and the numbers, the student received 0.1 flexibility points. For changing the problem structure (changing relations between objects in the problem, changing operations), the student received 1 flexibility point. For transferring the problem to another context (conventional: for example, instead of motion problem—cooperative work problem; unconventional: for example, instead of given "magic figure" – inventing own "magic figure"), the student received 10 flexibility points. Table 2.c presents Flexibility Scoring Scheme for Problem Posing.

Table 2c: Flexibility Scoring Scheme (Problem Posing, Interviews)

For each posed problem	Giving another representation of the problem (changing the story and numbers) or not using the given idea	0.1
	Changing the structure of the problem (relations in the problem)	1
	Transferring problem (conventional or unconventional) to another context or posing an unconventional problem when initial problem is conventional	10

The more the posed problem differs from the given problem, the more flexible it is.

Originality

To evaluate originality, we examined the correctness and solvability of the posed problem (Silver & Cai, 1996) and its conventionality. Table 2.d presents the Originality Scoring Scheme for Problem Posing.

Table 2.d: Originality Scoring Scheme (Problem Posing, Interviews)

For each posed problem	Statement, non-mathematical problem or unsolvable (incorrect) mathematical problem	0.1
	Conventional solvable mathematical problem	1
	Unconventional solvable mathematical problem	10

Creativity

The creativity of a posed problem solution (Cr_i) is the product of the flexibility (Flx_i) and the originality (Or_i): $Cr_i = Flx_i \cdot Or_i$. The total creativity score (Cr) is the sum of the creativity scores for each posed problem: $Cr = \sum_{i=1}^{N} Flx_i \cdot Or_i$ (N = fluency).

3.4.4 Research Procedure

Table 3.a: Research Procedure

Instrument	Population	Purpose
Interview A	6 students	Qualitative analysis of the changes in students' creativity (see research questions)
Interview B		
Interview C		

Interviewing was conducted as shown in Table 3.a. In addition, all inquiry projects performed by the students were collected and analyzed.

3.5 Data Analysis

3.5.1 Framework

The projects were conducted under the teacher's guidance (the students were asked questions, but the teacher didn't suggest ideas immediately leading to the solution, but rather gave the students small hints). The students which performed a part of a given project submitted it to the teacher for review. Once a week at least each student met the teacher-supervisor to discuss the project.

3.5.2 Analysis of Interviews: Problem Solving

As mentioned above, six students were interviewed individually: Joseph, Ian, Sam, Daniel, Dan and Tim. All names are pseudonyms.
The score for each creativity component was given according to the scoring schemes introduced in previous section. Table 3.b presents final scores for fluency, flexibility, originality, and creativity of problem solving for each interview (detailed scores are presented in Appendix 25, Tables A25.1, A25.2, and A25.3).

Table 3.b: Evaluation of Creativity in Problem Solving in Individual Interviews

Interview / Student's Name	Interview A				Interview B				Interview C			
	fl	**flex**	**or**	**Cr**	**fl**	**flex**	**or**	**Cr**	**fl**	**flex**	**or**	**Cr**
Joseph	8	71	33.2	242	9	61	33.1	241	10	82	53.1	433
Ian	10	71.2	33.4	242.02	6	51	33	321	8	71	43.1	341
Sam	7	52	10.6	104.2	8	61.1	33.2	241.01	5	50	31.1	311
Daniel	8	61.1	32.3	232.01	7	61	42.2	313	11	51.5	31.7	312.05
Dan	6	60	22.2	222	4	40	21.1	211	7	42.1	40.3	311.11
Tim	5	41	10.4	103.1	7	61	23.2	223	7	61	51.1	421

The analysis of the students' problem solving during the interviews in the research of Braverman (2010) revealed the following four main types of changes in their' problem-solving creativity:

1) Regressive creativity type
2) Progressive creativity type
3) Progressive-regressive creativity type
4) Promising creativity type

The characteristics of creativity types were as follows:

Regressive creativity type is characterized by decrease of all creativity components displayed in tests and in interviews during the intervention, non-changing of the balance between the potential and available solution spaces.

Progressive creativity type is characterized by increase of all creativity components displayed in tests and in interviews during the intervention, transformation of a part of the potential solution space into the available solution space.

Progressive-regressive creativity type is characterized by increase of all creativity components displayed in tests and in interviews in the first half of intervention and decrease in the second half, increase of both potential and available solution spaces in the first half of the intervention and decrease of both potential and available solution spaces in the second half.

Promising creativity type is characterized by decrease of all creativity components displayed in interviews in the first half of intervention and increase in the second half, increase of potential solution space.

As we can see from the Table 3.b, in this research all students belong to progressive creativity type and all of them have very high creative scores (all components).

3.5.3 Analysis of Interviews: Problem Posing

Table 3.c presents the final scores for fluency, flexibility, originality, and creativity scores for problem posing per each.

Table 3.4a: Evaluation of Creativity for Problem Posing in Individual Interviews

Interview / Student's Name	Interview A				Interview B				Interview C			
	fl	**flex**	**or**	**Cr**	**fl**	**flex**	**or**	**Cr**	**fl**	**flex**	**or**	**Cr**
Joseph	5	2.3	14	11.3	5	12.2	22.1	111.11	4	11.2	21.1	110.11
Ian	4	3.1	13	12.1	5	3.2	23	21.2	6	12.3	15	102.3
Sam	3	0.3	12	1.2	4	2.2	13	11.2	4	21.1	22	201.1
Daniel	3	2.1	11.1	10.2	4	11.2	11.2	101.02	3	20.1	12	110.1
Dan	4	2.2	13	3.1	3	11.1	12	101.1	6	13.2	42	130.02
Tim	3	1.2	12	10.2	4	4	22	22	3	10.2	10.2	100.02

The analysis of the students' problem posing during the interviews in the research of Braverman (2010) showed that the problem posing fluency of most students decreased during the intervention; most students posed analogous problems (low scores on flexibility and originality).

In this research problem posing creativity grows during the intervention.

3.5.4 Performed inquiry projects

I. **Series of natural numbers arranged in a special way.**

Student got an initial problem:

In each of the expressions bellow choose "+" or "-"among two consecutive numbers in the expression so that the sum will be equal to zero (if possible; n is natural number).

a) $\pm 1 \pm 2 \pm 3 \pm ... \pm 12$

b) $\pm 1 \pm 2 \pm 3 \pm ... \pm 13$

c) $\pm 1 \pm 2 \pm 3 \pm ... \pm 14$

d) $\pm 1 \pm 2 \pm 3 \pm ... \pm 15$

e) $\pm 1 \pm 2 \pm 3 \pm ... \pm n$

Possible solutions: we have many different solutions for the given problem which can be produced by different grouping of the given numbers. For example, we can group them by "quarters" of consequent numbers (according to remainders of 4).

If we have n=4k terms, it is possible with combination (+ — — +) for each “quarter” of consequent numbers (because of k - (k + 1) - (k + 2) + (k + 3) = 0):

(+ 1 –2 – 3+4)+(+5– 6– 7+8)+(+9– 10– 11+12)=0

If n=4k+3 terms, it is possible with combination (+ + —) for the first three numbers and with combination (+ — — +) for each “quarter” of consequent numbers:

(+ 1 +2 – 3)+(+4– 5– 6+7)+(+8– 9– 10+11)+(+12– 13– 14+15)=0

If n=4k+2 or n=4k+1 it is impossible (because in both these cases sum of all numbers is an odd number).

After that the student got the main problem:

Put "+" or "-" in expression:

± 1 ± 2 ± 3 ±...± n so as to receive number N; $1 \le N \le n$

First case: $4K$ numbers in consequence

$\pm 1 \pm 2 \pm 3 \pm 4 ..... \pm 4K = N$

If N - odd number it is impossible because:

$S = \frac{(4K)\cdot(4K+1)}{2} = \frac{2K \cdot (4K+1)}{1}$ - sum of all numbers and S is even number. So it’s clear that it's impossible to set $\pm$ so as to receive 0 in equation $S - N = 0$.

If N - even number it is possible and we have two sub-cases:

1) $N = 4M$

We can nullify numbers 1, 2, 3 as: 1+2-3 or -1-2+3. After that we’ll keep N on the left side and we have in our “equation” just quarters of consequent numbers which can be nullified by already known combinations: (+ — — +) or (— + + —).

Example: 1+2-3 +4-5-6+7 **+8** +9-10-11+12 = 8

2) $N = 4M + 2$

We can nullify numbers 1, 2, 3 as: 1+2-3 or -1-2+3. After that we’ll keep two pairs of numbers near N (pair before N and pair after N) on the left side of our “equation” and we have in our “equation” just quarters of consequent numbers, which can be nullified by already known combinations: (+ — — +) or (— + + —).

After that we’ll put the signs in the next way: + — N — + .

Example: 1+2-3 +4-5 **+6** -7+8 +9-10-11+12 = 6

Also the student found solution for cases $4K+1$, $4K+2$, $4K+3$ and special case N=2.

II. Optimization: balls' problem.

In a given inquiry project student took a famous math problem: given two balls and 100-floor building, the goal is to find by minimal number of throws a highest floor so, that if you will throw ball from there, it will not break.

The student made generalization of the problem and found the formulas:

$$\max(b,t) = t + \sum_{i=1}^{t-1} \max(b-1, i)$$, (b-number of balls, t-number of floors) and

$$\max(b,t) = \sum_{i=1}^{b} \binom{t}{i}$$, with binomial coefficients.

With t big enough we have: $\max(b,t) \approx \frac{t^b}{b!}$

III. What is behind the Ramsey Theorem? Solution of complex geometric problems using Ramsey Theory

Student created a new set of tools that use the principles of Ramsey Theorem, whereby geometric problems can be solved with the principles of the order of the content. It should be noted that the tools developed are new and have no mention in the literature. Research is very important in engineering and technology using geometry and in the Natural Sciences in genetics. There are several open questions, e.g. using of the method not in plane.

IV. Solution of symmetrical inequalities using majorants

In this research, a new revolutionary method has been developed. This method, allows us to solve easily complicated symmetrical inequalities. This distinctive method simplifies the proof of those inequalities, and its complication doesn't cross the regular primary level of math.

Symmetrical inequalities are algebraic expressions where each pair of variables can be exchanged, while preserving their algebraic meaning. For example: $a+b>0$ ($b+a>0$) or $a \cdot b > 0$ ($b \cdot a > 0$)

If $x>5$ then it's obvious that $x>3$, therefore $x>5$ is a "stronger" inequality than $x>3$.

<u>Solution of a symmetrical inequality with the new method</u>

Given: $x+y+z=0$. Prove: $x^2y^2+y^2z^2+x^2z^2+3\geq 6xyz$

At first, let's change the inequality:

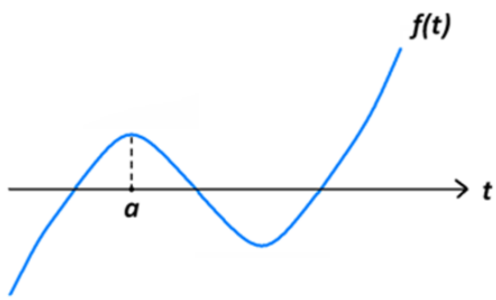
$$(x+y+z)^2 = x^2y^2+y^2z^2+x^2z^2+2x^2yz+2xy^2z+2xyz^2 =$$
$$= x^2y^2+y^2z^2+x^2z^2+2xyz(x+y+z)=x^2y^2+y^2z^2+x^2z^2$$

From here, we have to prove now an inequality:

$(x+y+z)^2+3\geq 6xyz$.

Now let's look at the polynomial:

$$f(t)=(t-x)(t-y)(t-z)=t^3-t^2(x+y+z)+t(xy+yz+xz)-xyz$$



The point a is found between the smallest root and the middle root, where the polynomial has a local maximum.

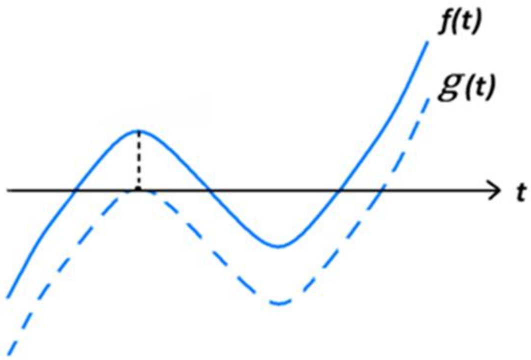
Let's look also at the polynomial:

$$g(t)=(t-a)(t-a)(t+2a)=t^3-t^2(a+a-2a)+t(aa-2aa-2aa)-aa(-2a)$$

Since $f'(a)=g'(a)$, therefore all coefficients of two polynomials are equal.

So, $aa-2aa-2aa-aa(-2a)=xe+yz+xz$

This way, the function g(t) is derived from the function f(t) by a movement on the Y axis. In addition, at the point a the value of f(t) is positive and the value of g(t) is equal to zero. From here, we can say that the movement on the Y axis was made downwards. In other words: $g(a)\leq f(a)$, therefore: $aa(-2a)\geq xyz$.



Now, let's refer to the inequality with the majorant that we found by the movement process, and with the roots of the polynomial g(t).

$$x^2y^2+y^2z^2+x^2z^2+3\geq 6xyz$$

$$(x+y+z)^2+3\geq 6xyz$$

$$(aa-2aa-2aa)^2+3\geq -12a^3$$

This inequality is "stronger" than the previous one. Therefore, it is enough to prove that:

$$9a^4+12a^3+3\geq 0.$$

$$9a^4+12a^3+3=(9a^2-6a+3)(a^2+2a+1)=((3a-1)^2+2)(a+1)^2\geq 0$$

V. Markov Diophantine equations

This project inquires Markov theorems, Markov equation $x^2+y^2+z^2=3xyz$ and using it for finding division rules for 3, 4, working with Markov triplets etc.

VI. Fake coins problems and their expansions for problems with more fake coins lying following to each other: finding algorithms

Here introduced three initial problems:

A) There are four similar-looking coins arranged in one row. There are two heavy fake coins. Is it possible to find these two fake coins, if it is known that they are next to each other? You can only use scales without weights. How many weighings do you need? What is the minimal number of weighings?

B) There are 10 similar-looking coins arranged in one row. There are two heavy fake coins. Is it possible to find these two fake coins if it is known that they are next to each other? You can only use scales without weights. How many weighings do you need? What is the minimal number of weighings?

C) There are 28 similar-looking coins arranged in one row. There are two heavy fake coins. Is it possible to find these two fake coins if it is known that they are next to each other? You can only use scales without weights. How many weighings do you need? What is the minimal number of weighings?

Problem A (solution with the minimal number of weighings):

"We put on the scales the two central coins, and then there are three possibilities:

a) The coins are equal; so, they both are fake;

b) The left coin weighs more; therefore, the two left coins are fake;

c) The right coin weighs more; therefore, the two right coins are fake.

So, we need only one attempt."

Problem B (solution with the minimal number of weighings):

"We can use the solution to Problem A. We need to divide the coins into three groups—3-4-3—and to put the two "triplets" on the scales. Then, we'll have the following options:

a) If they are not equal, we'll add to the "triplet" that weighs more the next coin from the "quadruplet." Then, we'll continue according to Problem A (the fake coins are among these four coins).

b) If two "triplets" are equal, we'll continue with the "quadruplet" according to problem A (the fake coins are among these four coins).

So, we need 2 weighings."

Problem C (solution with more than a minimal number of weighings):

"We can use the solution to Problem B. We need to divide coins into three groups—9-10-9—and to put two "nines" on the scales; then, we'll have following options:

a) If there is no equality, we'll add to the nine that weigh more one coin nearest to this nine from the ten. Then we'll continue according to Problem B (the fake coins are among these ten coins).

b) If two nines are equal, we'll continue with the ten according to Problem B (the fake coins are among these ten coins).

So, we need 3 weighings."
After solution of three previous problems one can do generalization. After 28 we have 27+28+27=82 coins and 4 weighings. The formula is: $3^m + k - 1$ (m-number of weighings, k-number of fake coins).

3.6 Contribution and Limitations of the Study

3.6.1 Contribution of the Study

We found that, consistently with findings of other studies (e.g. Levav-Weinberg & Leikin, 2009, Braverman, 2010) this research showed that didactical tools (inquiry projects) lead to the development of creativity. The study develops a set of inquiry-based projects that can be used by teachers in mathematics classes of different levels (especially with high-achievers in mathematics). These projects can also be used in professional development courses for teachers of mathematics in order to intensify the implementation of inquiry-based approach in schools. It is important to stress, that students which participated in the study, not only high-ability students, their creativity scores were significantly high during all the intervention.

3.6.2 Limitations of the Study

There were the following limitations to this study:
We do not use a control group in the study. Small number of students participated in the study and we do not compare them with regular students.
2) The study leaves open a question about the way to raise students' motivation to participate in inquiry projects.

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